

Prove that $\sum_{i=1}^n (2i-3)3^{i-1} = (n-2)3^n + 2$ for all positive integers n using mathematical induction.

SCORE: ____ / 15 PTS

Basis case:
$$\sum_{i=1}^1 (2i-3)3^{i-1} = (-1)3^0 = -1 = (-1)3^1 + 2$$

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Inductive step: Assume that $\sum_{i=1}^k (2i-3)3^{i-1} = (k-2)3^k + 2$ for some arbitrary integer $k \geq 1$

Prove that
$$\sum_{i=1}^{k+1} (2i-3)3^{i-1} = (k+1-2)3^{k+1} + 2 = (k-1)3^{k+1} + 2$$

$$\begin{aligned} & \sum_{i=1}^{k+1} (2i-3)3^{i-1} \\ &= \sum_{i=1}^k (2i-3)3^{i-1} + (2(k+1)-3)3^k \\ &= \sum_{i=1}^k (2i-3)3^{i-1} + (2k-1)3^k \\ &= (k-2)3^k + 2 + (2k-1)3^k \\ &= (k-2)3^k + (2k-1)3^k + 2 \\ &= (k-2+2k-1)3^k + 2 \\ &= (3k-3)3^k + 2 \\ &= 3(k-1)3^k + 2 \\ &= (k-1)3^{k+1} + 2 \end{aligned}$$

So, by mathematical induction, $\sum_{i=1}^n (2i-3)3^{i-1} = (n-2)3^n + 2$ for all positive integers n

Expand and simplify $(\sqrt{t} - 2t^3)^4$.

SCORE: _____ / 6 PTS

You may write the coefficients in your final answer in factored form, as shown in lecture.

The coefficients in your final answer must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

NOTE: Do NOT use your calculator's ! nor $C(n, r)$ features.

$$\begin{aligned} & (\sqrt{t})^4 + \overset{\textcircled{1}}{4}(\sqrt{t})^3(-2t^3) + \overset{\textcircled{1}}{6}(\sqrt{t})^2(-2t^3)^2 + \overset{\textcircled{1}}{4}(\sqrt{t})(-2t^3)^3 + (-2t^3)^4 \\ & = t^2 - 8t^{\frac{9}{2}} + 24t^7 - 32t^{\frac{19}{2}} + 16t^{12} \\ & \quad \underbrace{\textcircled{\frac{1}{2}} \quad \textcircled{\frac{1}{2}} \quad \textcircled{\frac{1}{2}} \quad \textcircled{\frac{1}{2}} \quad \textcircled{1}} \end{aligned}$$

Consider the expansion of $(7x^{12} + \frac{2}{x^8})^{20}$.

SCORE: ____ / 9 PTS

For the questions below, you may write the coefficients in your final answers in factored form, as shown in lecture.

The coefficients in your final answers must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

NOTE: Do NOT use your calculator's ! nor $C(n, r)$ features.

[a] Find the coefficient of x^{10} in the expansion.

$\frac{1}{2}$ POINT EACH EXCEPT

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$$\sum_{r=0}^{20} \binom{20}{r} (7x^{12})^{20-r} \left(\frac{2}{x^8}\right)^r = \sum_{r=0}^{20} \binom{20}{r} 7^{20-r} x^{12(20-r)} 2^r x^{-8r} = \sum_{r=0}^{20} \binom{20}{r} 7^{20-r} 2^r x^{240-20r}$$
 AS NOTED

$$x^{240-20r} = x^{10} \Rightarrow 240 - 20r = 10 \Rightarrow 24 - 2r = 1 \Rightarrow r = \frac{23}{2} \text{ which is not an integer}$$

No x^{10} term, so coefficient = 0

[b] Find the coefficient of x^{-80} in the expansion.

$$x^{240-20r} = x^{-80} \Rightarrow 240 - 20r = -80 \Rightarrow 24 - 2r = -8 \Rightarrow r = 16$$

$$\binom{20}{16} 7^{20-16} 2^{16} = \binom{20}{16} 7^4 2^{16} = \frac{20!}{16!4!} 7^4 2^{16} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!}{16! \cdot 4 \cdot 3 \cdot 2 \cdot 1} 7^4 2^{16} = 5 \cdot 19 \cdot 3 \cdot 17 \cdot 7^4 2^{16}$$

[c] Find the sixth term in the expansion.

$$\binom{20}{5} (7x^{12})^{20-5} \left(\frac{2}{x^8}\right)^5 = \frac{20!}{5!15!} 7^{15} x^{180} 2^5 x^{-40} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 15!} 7^{15} 2^5 x^{140}$$

$$= 19 \cdot 3 \cdot 17 \cdot 16 \cdot 7^{15} 2^5 x^{140}$$